On approximate controllability of nonlocal impulsive fractional stochastic differential equations of order $q \in (1, 2]$ with infinite delay and Poisson jumps in Hilbert space

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Abstract. In this work, we established the approximate controllability of second-order ($q \in (1, 2]$) nonlocal impulsive fractional stochastic differential equations with infinite delay and Poisson jumps in Hilbert space.The approximate controllability of the established control system is evaluated by using the q-order cosine family of operators, stochastic analysis techniques, a new set of sufficient conditions are derived for the approximate controllability. Finally an application is added to illustrate the main theory.

Key Words : Approximate controllability, fractional stochastic differential systems, Hilbert space, Poisson jumps.

MSC 2010: 34K50, 35A01, 34K50, 93E03.

1. Introduction

During the last few decades, fractional differential equations and their applications, we refer to the monographs [20, 23, 30, 37] and references cited therein have occupied a lot of important field that has become a powerful tool in modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering [4-11]. Controllability plays a very important role in various areas of engineering and science. In particular control systems contains many fundamental problems of control theory such as optimal control, stability can be solved with assumption that the system is controllable [3, 22]. In the infinite dimensional systems, two basic concepts of controllability are exact and approximate controllability. This is strongly related to fact that in infinite dimensional spaces there exist linear subspaces, which are not closed. Exact controllability enables to steer the system to arbitrary final state while approximate controllability is weaker concept of controllability and it is possible to steer the system to an arbitrary small neighborhood of the final state (see, for

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example, [4, 12, 26, 42, 46]). Second order integer and fractional order differential systems with impulses has received more and more attention very recently, see the monographs of Lakshmikantham et al. [21], the papers [2, 15, 24, 29, 39], and the references therein related to this matter. A natural extension of a deterministic differential equation model is a system of stochastic differential equations (see the monographs of Astrom [1]), where relevant parameters are modeled as suitable stochastic processes. This is due to the fact that most problems in real-life situations are basically modeled by stochastic equations (see [9, 12, 13, 45]) rather than deterministic [10, 19, 36, 37]. The existence and controllability of second order integer and fractional order stochastic differential equations in infinite dimensional spaces have been established by many authors [13, 16, 25, 33, 43, 44]. Kexue et al. [19] studied controllability of nonlocal fractional differential systems of order $\alpha \in (1, 2]$ in Banach spaces. Muthukumar and Thiagu [27] proved the existence of solutions and approximate controllability of fractional nonlocal stochastic differential equations of order $q \in (1, 2]$ with infinite delay and Poisson jumps in Hilbert spaces by using fixed point theory and natural assumption that the corresponding linear system is approximately controllable. Hence the problem of existence of solutions and approximate controllability for nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delays has been studied by several authors were received significant attention in modern days (see [9, 10, 12, 35, 40] and references therein). Moreover, fractional stochastic differential equations with Poisson jumps are discussed by many authors [27, 34, 38, 41].

Motivated by [19, 27, 28, 33], we address the sufficient conditions for ensuring the approximate controllability of second-order $(1 < q \leq 2)$ nonlocal impulsive fractional stochastic differential equations with infinite delay and Poisson jumps in Hilbert space described by

$$\begin{aligned} {}^cD_t^q x(t) &= Ax(t) + Bu(t) + f\Big(t, x_t, \int_0^t \sigma(t,s,x_s)ds\Big) + \int_{-\infty}^t g(\tau, x_\tau)d\omega(\tau) \\ &\quad + \int_Z \mathfrak{h}(t, x_t, \eta)\widetilde{N}(dt, d\eta), \quad t \in J := [0,b] \setminus \{t_1, ..., t_n\} \\ x_0(t) &= \phi(t) + \mathbf{m}(x_{t_1}, ..., x_{t_m})(t), \quad t \in (-\infty, 0], \\ x'(0) + \mathbf{n}(x) &= \xi, \\ \Delta x(t_k) &= I_k(x_{t_k}), \\ \Delta x'(t_k) &= \overline{I}_k(x_{t_k}), \quad k = 1, 2, ..., n =: \overline{1, n}, \end{aligned} \tag{1}$$

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Here, the state variable $x(\cdot)$ takes values in a real separable Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_H$. The fractional derivative ${}^cD_t^q$, $1 < q \leq 2$ is understood in the Caputo sense. Let $0 = t_0 < t_1 < t_2 < ... < t_n < b$ be the given time points. Also $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous q-order cosine family $C_q(t)$ on H. Let K be another separable Hilbert space. Let $\{\omega(t)\}_{t\geq 0}$ be a given K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. Let $\widetilde{q} = \{\widetilde{q}(t) : t \in D_{\widetilde{q}}\}$ be a stationary $\mathfrak{F}_t$ Poisson point process with characteristic measure λ. Let $N(dt, d\eta)$ be the Poisson counting measure associated with $\widetilde{q}$. Then $N(t, Z) = \sum_{s\in D_{\widetilde{q}}, s\leq t} I_Z(\widetilde{q}(s))$ with measurable set $Z \in \widetilde{B}(K - \{0\})$, which denotes the Borel σ-field of $K - \{0\}$. Let $\widetilde{N}(dt, d\eta) = N(dt, d\eta) - dt\lambda(d\eta)$ be the compensated Poisson measure that is independent of $\omega(t)$. Let $P_2([0, b] \times Z; H)$ be the space all mapping $\chi : [0, b] \times Z \to H$ for which $\int_0^b \int_Z \mathbb{E}\|\chi(t, \eta)\|_H^2 dt\lambda(d\eta) < \infty$. We can define the H-valued stochastic integral $\int_0^b \int_Z \chi(t, \eta)\widetilde{N}(dt, d\eta)$, which is a centred square integrable martingale. We can also employing the same notation $\|\cdot\|$ for the norm of $\mathcal{L}(K, H)$, which denotes the space of all bounded operators from K into H. Simply $\mathcal{L}(H)$ if $K = H$. The histories x_t represents the function defined by $x_t : (-\infty, 0] \to H$, $x_t(\theta) = x(t+\theta)$, for $t \geq 0$ belong to some phase space $\mathcal{B}$ defined axiomatically. Further $\sigma : J\times J\times\mathcal{B} \to H$, $f : J\times\mathcal{B}\times H \to H$, $g : J \times \mathcal{B} \to \mathcal{L}_Q(K, H)$ and $\mathfrak{h} : J \times \mathcal{B} \times Z \to H$ are nonlinear functions. Here $\mathcal{L}_Q(K, H)$ denotes the space of all Q-Hilbert Schmidt operators from K into H. Moreover the function $\mathbf{m} : H^m \to H$ and and $\mathbf{n} : H \to H$,where $0 = t_0 < t_1 < t_2 < ... < t_m \leq T$, $m \in \mathbb{N}$ are continuous functions. For example

$$\mathbf{m}(x_{t_1}, ..., x_{t_m})(t) = \sum_{i=0}^{m} C_i z(t_i + t), x \in \mathcal{B}, t \in (-\infty, 0],$$

where $C_i(i = 1, 2, ..., m)$ are constants. I_k and $\overline{I}_k : H \to H$ are appropriate functions. The symbol $\Delta\zeta(t)$ represents the jump of the function ζ at t which is defined by $\Delta\zeta(t) = \zeta(t^+) - \zeta(t^-)$. The initial data $\phi = \{\phi(t) : t \in (-\infty, 0]\}$ is an $\mathfrak{F}_0$-measurable $\mathcal{B}$-valued stochastic process independent of Brownian motion $\{\omega(t)\}$ and Poisson point process $\widetilde{q}(\cdot)$ with finite second moment. Further $\xi(t)$ is an $\mathfrak{F}_t$-measurable H-valued random variable independent of $\omega(t)$ and Poisson point process $\widetilde{q}$ with finite second moment. In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato [17]. The axioms of the space $\mathcal{B}$ are established for $\mathcal{F}_0$- measurable functions from J_0 into H, where $J_0 := (-\infty, 0]$, endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and satisfying the following axioms:

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(a) If $x : (-\infty, b) \to H, b > 0$ is continuous on $[0, b)$ and $x_0 \in \mathcal{B}$, then for each $t \in [0, b)$ the following conditions hold,

  i) $x_t \in \mathcal{B}$,

  ii) $\|x(t)\| \leq K_1 \|x_t\|_{\mathcal{B}}$,

  iii) $\|x_t\|_{\mathcal{B}} \leq K_2(t)\|x_0\|_{\mathcal{B}} + K_3(t) \sup\{\|x(s)\|; 0 \leq s \leq b\}$ where $K_1 > 0$ is a constant, $K_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally bounded function, $K_3 : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Moreover $K_1, K_2(\cdot), K_3(\cdot)$ are independent of $x(\cdot)$,

(b) For the function $x(\cdot)$ in (a), x_t is a $\mathcal{B}$- valued continuous functions on $[0, b)$,

(c) The space $\mathcal{B}$ is complete.

Example 1.1. *Let $r \geq 0$. $1 \leq q_1 < \infty$ and let $\widetilde{h} : (-\infty, r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions $(H5) - (H12)$ in the terminology of Hino et al.* [18]. *Briefly, this means that $\widetilde{h}$ is locally integrable and there is a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $\widetilde{h}(\xi_1 + \tau_1) \leq \gamma(\xi_1)\widetilde{h}(\tau_1)$ for all $\xi_1 \leq 0$ and $\tau_1 \in (-\infty, -r) \setminus \mathcal{N}_{\xi_1}$, where $\mathcal{N}_{\xi_1} \subseteq (-\infty, -r)$is a set whose Lebesgue measure zero. We denote by $\mathcal{C}_r \times \mathcal{L}_{q_1}(\widetilde{h}, H)$ the set consists of all classes of functions $\varphi : (-\infty, 0] \to H$ such that $\varphi|_{[-r,0]} \in \mathcal{C}([-r, 0], H)$, $\varphi(\cdot)$ is Lebesgue measurable on $(-\infty, -r)$, and $\widetilde{h}\|\varphi\|^{q_1}$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm is given by*

$$\|\varphi\|_{\mathcal{B}} = \sup_{-r \leq \tau_1 \leq 0} \|\varphi(\tau_1)\| + \left(\int_{-\infty}^{-r} \widetilde{h}(\tau_1)\|\varphi(\tau_1)\|^{q_1} d\tau_1 \right)^{\frac{1}{q_1}}$$

The space $\mathcal{C}_r \times \mathcal{L}_{q_1}(\widetilde{h}, H)$ satisfies axioms (a)-(c). Moreover when $r = 0$ and $q_1 = 2$, We can write $K_1 = 0$, $K_2(t) = \gamma(-t)^{\frac{1}{2}}$ and $K_3(t) = 1 + \left(\int_{-t}^{0} \widetilde{h}(\tau_1) d\tau_1 \right)^{\frac{1}{2}}$ for $t \geq 0$ (see [18]*).*

The $\mathcal{B}$- valued stochastic process $x_t : \Omega \to \mathcal{B}, t \geq 0$, is defined by setting

$$x_t = \{x(t + s)(w) : s \in (-\infty, 0]\}.$$

The collection of all strongly measurable, square integrable H valued random variables, denoted by $\mathcal{L}_2(\Omega, \mathfrak{F}, \mathbb{P}; H) \equiv \mathcal{L}_2(\Omega; H)$, is a Banach space equipped with norm $\|x(\cdot)\|_{\mathcal{L}_2} = (\mathbb{E}\|x(\cdot; w)\|_H^2)^{\frac{1}{2}}$, where the expectation, $\mathbb{E}$ is defined by $\mathbb{E}(h) = \int_\Omega h(w) d\mathbb{P}$. Let $J_1 = (-\infty, b]$ and $C(J_1, \mathcal{L}_2(\Omega; H))$ be the Banach space of all continuous maps from J_1 into $\mathcal{L}_2(\Omega; H)$ satisfying the condition $\sup_{t \in J_1} \mathbb{E}\|x(t)\|^2 < \infty$.

Let $\mathbb{Z}$ be the closed subspace of all continuous process x that belong to the space $C(J_1, \mathcal{L}_2(\Omega; H))$ consisting of $\mathfrak{F}_t$- adapted measurable process and $\mathfrak{F}_o$- adapted processes $\phi, \xi \in \mathcal{L}_2(\Omega, \mathcal{B})$ and the

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restriction $x : J \to \mathcal{L}_2(\Omega, \mathcal{B})$ is continuous. Let $\| \cdot \|_{\mathbb{Z}}$ be a seminorm in $\mathbb{Z}$ defined by

$$\|x\|_{\mathbb{Z}}^2 \quad = \quad \sup_{t \in J} \|x_t\|_{\mathcal{B}}^2,$$

where,

$$\|x_t\|_{\mathcal{B}}^2 \leq \overline{K_2} \mathbb{E} \|\phi\|_{\mathcal{B}}^2 + \overline{K_3} \sup_{0 \leq s \leq b} \left\{ \mathbb{E} \|x(s)\|^2 \right\},$$

$\overline{K_2} = \sup_{t \in J} \{K_2(t)\}$ and $\overline{K_3} = \sup_{t \in J} \{K_3(t)\}$. It is easy to verify that $\mathbb{Z}$ furnished with the norm topology as defined above is a Banach space.

2. Preliminaries

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\mathfrak{F}_t : t \in J\}$ satisfying $\mathfrak{F}_t \subset \mathfrak{F}$. An H-valued random variable is an $\mathfrak{F}$-measurable function $x(t) : \Omega \to H$ and a collection of random variables $S = \{x(t, w) : \Omega \to H|_{t \in J}\}$ is called a stochastic process. Usually, we suppress the dependence on $w \in \Omega$ and write $x(t)$ instead of $x(t, w)$ and $x(t) : J \to H$ in the place of S. Let $\beta_n(t)$ $(n = 1, 2, \ldots)$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent over $(\Omega, \mathfrak{F}, \mathbb{P})$. Set $\omega(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n, t \geq 0$, where $\lambda_n \geq 0$ $(n = 1, 2, \ldots)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2, \ldots)$ is complete orthonormal basis in K. Let $Q \in \mathcal{L}(K, K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite $Tr(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$, $Tr(Q)$ denotes the trace of the operator Q. Then the above K-valued stochastic process $\omega(t)$ is called a Q-Wiener process. Let us assume that $\mathfrak{F}_t = \sigma(\omega(s) : 0 \leq s \leq t)$ is the σ-algebra generated by ω and $\mathfrak{F}_t = \mathfrak{F}$. Let $\varphi \in \mathcal{L}(K, H)$ and define $\|\varphi\|_Q^2 = Tr(\varphi Q \varphi^\star) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \varphi e_n\|^2$. If $\|\varphi\|_Q < \infty$, then φ is called a Q-Hilbert Schmidt operator. Let $\mathcal{L}_Q(K, H)$ denotes the space of all Q-Hilbert Schmidt operators $\varphi : K \to H$. The completion $\mathcal{L}_Q(K, H)$ of $\mathcal{L}(K, H)$ with respect to the topology induced by a norm $\| \cdot \|_Q$, where $\|\varphi\|_Q^2 = \langle \varphi, \varphi \rangle$ is a Hilbert space with the above norm topology. Let $\mathcal{C}$ be the closed subspace of all continuously differentiable process that belong to the space $\mathcal{C}(J, \mathcal{L}_2(\Omega; H))$ consisting of $\mathfrak{F}_t$-adapted measurable process such that $\mathfrak{F}_0$-adapted processes ϕ, $\xi \in \mathcal{L}_2(\Omega, \mathcal{B})$.

Let $\mathcal{B}(H)$ be the space of all bounded linear operators on H. Let I be the identity operator on H. If A is a linear operator on H, then $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A. We can use the notation

$$k_\beta(t) = \frac{t^{\beta - 1}}{\Gamma(\beta)}, \quad t > 0, \quad \beta > 0$$

where $\Gamma(\beta)$ is the Gamma function. If $\beta = 0$, we set $k_0(t) = \delta(t)$, the delta distribution.

Definition 2.1. [19] *The Riemann-Liouville fractional integral of order $q > 0$ is defined by*

$$\mathcal{J}_t^q x(t) = \int_0^t k_q(t-s)x(s)ds,$$

where $x(t) \in \mathcal{L}_1([0,b];H)$.

Definition 2.2. [19] *The Riemann-Liouville fractional derivative of order $1 < q \leq 2$ is defined by*

$$D_t^q x(t) = \frac{d^2}{dt^2}\mathcal{J}_t^{2-q}x(t),$$

where $x(t) \in \mathcal{L}_1([0,b];H)$, $D_t^q x(t) \in \mathcal{L}_1([0,b];H)$.

Definition 2.3. [19] *The Caputo fractional derivative of order $1 < q \leq 2$ is defined by*

$${}^cD_t^q x(t) = D_t^q(x(t) - x(0) - x'(0)t),$$

where $x(t) \in \mathcal{L}_1([0,b];H) \cap \mathcal{C}^1([0,b];H)$, $D_t^q x(t) \in \mathcal{L}_1([0,b];H)$.

The Laplace transform for the Riemann-Liouville fractional integral is given by

$$L[\mathcal{J}_t^q x(t)] = \frac{1}{\lambda^q}x_L(\lambda)$$

where $x_L(\lambda)$ is the Laplace transform of x given by

$$x_L(\lambda) = \int_0^\infty e^{-\lambda t}x(t)dt, \quad Re\lambda > \omega_0.$$

The Laplace transform of the Caputo derivative is given by

$$L[{}^cD_t^q x(t)] = \lambda^q x_L(\lambda) - x(0)\lambda^{q-1} - x'(0)\lambda^{q-2}$$

Consider a class of fractional differential system with infinite delay

$$\begin{aligned} &{}^cD_t^q x(t) = Ax(t) + Bu(t), \quad t \in [0,b], \\ &x(0) = x_0 \in H, \\ &x'(0) = x_1 \in H, \end{aligned} \tag{2}$$

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where $q \in (1,2]$, ${}^cD_t^q$ is the Caputo fractional derivative, A is the infinitesimal generator of a strongly continuous q-order cosine family $\{C_q(t)\}_{t\geq 0}$ on the Hilbert space H, the state values $x(\cdot)$ takes values in H, the control function $u(\cdot)$ is given in $\mathcal{L}_2(J,U)$ of admissible control functions with U, a Hilbert space, B is a bounded linear operator form U into H.

Definition 2.4. [14] *The infinitesimal generator $A : H \to H$ of $\{C(t) : t \in \mathbb{R}\}$ is given by $Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}$ for all $x \in D(A) = \{x \in H : C(\cdot)x \in C^2(\mathbb{R}; H)\}$. It is known that the infinitesimal generator A is closed, densely defined operator on H for cosine and corresponding sine families and their generators.*

Definition 2.5. [19] *Let $q \in (1,2]$. A family $\{C_q(t)\}_{t\geq 0} \subset \mathcal{B}(H)$ is called a solution operator (or a strongly continuous q-order fractional cosine family) for the above problem (2) if the following conditions are satisfied*

(a) *$C_q(0) = I$, I is the identity operator in H,*

(b) *$C_q(t)x$ is strongly continuous for $t \geq 0$, for every $x \in H$,*

(c) *$C_q(t)D(A) \subset D(A)$ and $AC_q(t)x_0 = C_q(t)Ax_0$ for all $x_0 \in D(A)$, $\quad t \geq 0$,*

(d) *$C_q(t)x_0$ is a solution of $x(t) = x_0 + \int_0^t k_q(t-s)Ax(s)ds$ for all $x_0 \in D(A)$, $\quad t \geq 0$.*

A is called the infinitesimal generator of $C_q(t)$. The strongly continuous q-order fractional cosine family is also called q-order cosine family.

The corresponding fractional sine family $S_q : \mathbb{R}_+ \to \mathcal{B}(H)$ associated with C_q is defined by $S_q(t)x = \int\limits_0^t C_q(s)xds$ for $t \geq 0$, $x \in H$.

Definition 2.6. *The fractional Riemann-Liouville family $T_q : \mathbb{R}_+ \to \mathcal{B}(H)$ associated with C_q is defined by*

$$T_q(t) = \mathcal{J}_t^{q-1}C_q(t).$$

Definition 2.7. *The q-order cosine family C_q is called exponentially bounded if there are constants $M \geq 1$ and $\omega_0 \geq 0$ such that*

$$\|C_q(t)\| \leq Me^{\omega_0 t}, \quad t \geq 0.$$

An operator A is said to belong to $C^q(M, \omega_0)$, if the problem (2) has an q - order cosine family $C_q(t)$ satisfying the above inequality.

Assume $A \in C^q(M, \omega_0)$ and let $C_q(t)$ be the corresponding q-order cosine family. Then we have

$$\lambda^{q-1}R(\lambda^q, A)x_0 = \int_0^\infty e^{-\lambda t}C_q(t)x_0 dt, \quad Re\lambda > \omega_0, \quad x_0 \in H,$$
$$\lambda^{q-2}R(\lambda^q, A)x_0 = \int_0^\infty e^{-\lambda t}S_q(t)x_0 dt, \quad Re\lambda > \omega_0, \quad x_0 \in H,$$
$$R(\lambda^q, A)x_0 = \int_0^\infty e^{-\lambda t}T_q(t)x_0 dt, \quad Re\lambda > \omega_0, \quad x_0 \in H.$$

Based on the result in [19], the function $x \in \mathcal{C}([0,b]; H)$ is called a mild solution of (2) if x satisfies

$$x(t) = C_q(t)x_0 + S_q(t)x_1 + \int_0^t T_q(t-s)Bu(s)ds, \quad t \in J.$$

Also consider the linear equation

$$\begin{aligned} {}^cD_t^q x(t) &= Ax(t) + Bu(t) + f(t) + \int_{-\infty}^t g(t)d\omega(t) + \int_Z \mathfrak{h}(t)\widetilde{N}(dt, d\eta), \\ &\quad + \sum_{0<t_k<t} I_k(x_{t_k}) + \sum_{0<t_k<t} \overline{I}_k(x_{t_k}), k = \overline{1,n}, t \in J := [0,b] \setminus \{t_1, ..., t_n\} \\ x_0(t) &= \phi(t) + \mathbf{m}(x_{t_1}, ..., x_{t_m})(t), \quad t \in (-\infty, 0], \\ x'(0) &+ \mathbf{n}(x) = \xi. \end{aligned}$$

Taking Laplace transform, we get

$$\begin{aligned} &\lambda^q\widehat{x}(\lambda) - x(0)\lambda^{q-1} - x'(0)\lambda^{q-2} \\ &= A\widehat{x}(\lambda) + B\widehat{u}(\lambda) + \widehat{f}(\lambda) + \int_{-\infty}^t \widehat{g}(\lambda)d\omega(\lambda) + \int_Z \widehat{\mathfrak{h}}(\lambda)\widetilde{N}(d\lambda, d\eta) \\ &\quad + \sum_{0<t_k<t} \widehat{I}_k(x_{\lambda_k}) + \sum_{0<t_k<t} \widehat{\overline{I}}_k(x_{\lambda_k}), k = \overline{1,n}, \end{aligned}$$

where $\widehat{x}(\lambda)$, $\widehat{u}(\lambda)$, $\widehat{f}(\lambda)$, $\widehat{g}(\lambda)$, $\widehat{\mathfrak{h}}(\lambda)$, $\widehat{I}_k$ and $\widehat{\overline{I}}_k$ are the Laplace transform of $x(t)$, $u(t)$, $f(t)$, $g(t)$, $\mathfrak{h}(t)$, I_k and $\overline{I}_k$ respectively. Then we have

$$\begin{aligned} \lambda^q\widehat{x}(\lambda) &= x(0)\lambda^{q-1} + x'(0)\lambda^{q-2} + A\widehat{x}(\lambda) + B\widehat{u}(\lambda) + \widehat{f}(\lambda) \\ &\quad + \int_{-\infty}^t \widehat{g}(\lambda)d\omega(\lambda) + \int_Z \widehat{\mathfrak{h}}(\lambda)\widetilde{N}(d\lambda, d\eta) \\ &\quad + \sum_{0<t_k<t} \widehat{I}_k(x_{\lambda_k}) + \sum_{0<t_k<t} \widehat{\overline{I}}_k(x_{\lambda_k}), k = \overline{1,n}, \end{aligned}$$

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Again we can be written as

$$\begin{aligned} x(t) = {} & x(0)R(\lambda^q, A)\lambda^{q-1} + x'(0)R(\lambda^q, A)\lambda^{q-2} \\ & + B\widehat{u}(\lambda)R(\lambda^q, A) + \widehat{f}(\lambda)R(\lambda^q, A) \\ & + \int_{-\infty}^{t} R(\lambda^q, A)\widehat{g}(\lambda)d\omega(\lambda) + \int_Z \widehat{\mathfrak{h}}(\lambda)R(\lambda^q, A)\widetilde{N}(d\lambda, d\eta) \\ & + \sum_{0<t_k<t} R(\lambda^q, A)\widehat{I}_k(x_{\lambda_k}) + \sum_{0<t_k<t} R(\lambda^q, A)\widehat{\overline{I}}_k(x_{\lambda_k}), k = \overline{1, n}. \end{aligned}$$

From the property of Laplace transform, we get

$$\begin{aligned} x(t) = {} & C_q(t)(\phi(0) + \mathbf{m}(x_{t_1}, ..., x_{t_m})(0)) + S_q(t)(\xi - \mathbf{n}(x)) \\ & + \int_0^t T_q(t-s)Bu(s)ds + \int_0^t T_q(t-s)f(s)ds \\ & + \int_0^t T_q(t-s)\Big[\int_{-\infty}^{t} g(\tau)d\omega(\tau)\Big]ds \\ & + \int_0^t \int_Z T_q(t-s)\mathfrak{h}(s)\widetilde{N}(ds, d\eta) \\ & + \sum_{0<t_k<t} T_q(t-t_k)I_k(x_{t_k}) + \sum_{0<t_k<t} T_q(t-t_k)\overline{I}_k(x_{t_k}), k = \overline{1, n}. \end{aligned}$$

Definition 2.8. [27] *An H-valued stochastic process $\{x(t) : t \in (-\infty, b]\}$ is said to be a mild solution of the system (1) if*

(i) *$x(t)$ is $\mathfrak{F}_t$-adapted and measurable for $t \geq 0$,*

(ii) *$x(t)$ is continuous on $[0, b]$ almost surely and for each $s \in [0, t)$, the functions f, g and $\mathfrak{h}$ are integrable such that the following stochastic integral equation is satisfied*

$$\begin{aligned} x(t) = {} & C_q(t)\Big(\phi(0) + \mathbf{m}(x_{t_1}, ..., x_{t_m})(0)\Big) + S_q(t)(\xi - \mathbf{n}(x)) \\ & + \int_0^t T_q(t-s)f(s, x_s, \int_0^s \sigma(s, \tau, x_\tau)d\tau)ds + \int_0^t T_q(t-s)Bu(s)ds \\ & + \int_0^t T_q(t-s)\Big[\int_{-\infty}^{s} g(\tau, x_\tau)d\omega(\tau)\Big]ds \\ & + \int_0^t \int_Z T_q(t-s)\mathfrak{h}(s, x_s, \eta)\widetilde{N}(ds, d\eta) \end{aligned}$$

$$+\sum_{0<t_k<t} T_q(t-t_k)I_k(x_{t_k}) + \sum_{0<t_k<t} T_q(t-t_k)\overline{I}_k(x_{t_k}), k=\overline{1,n}.$$

(iii) $x_0(t)=\phi(t)+\mathbf{m}(x_{t_1},...,x_{t_m})(t)$, $t\in(-\infty,0]$, $x'(0)+\mathbf{n}(x)=\xi$.

In order to prove the theorems, we need the following hypotheses:

(H_1) A is an infinitesimal operator of a strongly continuous q-order cosine family $\{C_q(t):t\geq 0\}$ on H. Also $\|T_q(t)\|\leq M$, $\|C_q(t)\|\leq M_C$ and $\|S_q(t)\|\leq M_S$. Moreover the operators$T_q(t)$, $C_q(t)$ and $S_q(t)$ are compact for $t>0$.

(H_2) A is the infinitesimal generator of a strongly continuous q-order fractional cosine family $\{C_q(t)\}_{t\geq 0}$ on H.

(H_3) The operator B is the bounded linear operator from $\mathcal{L}_2(J;U)$ to $\mathcal{L}_2(J;H)$.

(H_4) The functions $\mathbf{m}:H^m\to H$ and $\mathbf{n}:H\to H$ are continuous, satisfy the Lipschitz conditions and there exists positive constant $M_{\mathbf{m}}$,$M_{\mathbf{n}}>0$ such that

$$\mathbb{E}\|\mathbf{m}(x)-\mathbf{m}(y)\|_H^2\leq M_{\mathbf{m}}\|x-y\|_{\mathcal{B}}^2$$
$$\mathbb{E}\|\mathbf{n}(x)-\mathbf{n}(y)\|_H^2\leq M_{\mathbf{n}}\|x-y\|_{\mathcal{B}}^2$$

for all x, $y\in\mathcal{B}$ and $t\in J$.

(H_5) The nonlinear functions f,g satisfies the Lipschitz condition and there exists positive constants $M_{f_1},M_{f_2},M_g>0$ such that

$$\|f(t,\varphi_1,x_1)-f(t,\varphi_2,x_2)\|^2\leq M_{f_1}\|\varphi_1-\varphi_2\|_{\mathcal{B}}^2+M_{f_2}\|x_1-x_2\|_H^2,$$
$$\|g(t,x_1)-g(t,x_2)\|^2\leq M_g\|x_1-x_2\|_{\mathcal{B}}^2,$$

for all $x_1,x_2,\varphi_1,\varphi_2\in\mathcal{B}$ and $t\in J$.

(H_6) For each $\varphi\in\mathcal{B}$, $K_1(t)=\lim\limits_{a\to\infty}\int\limits_{-a}^{0} g(s,\varphi)d\omega(s)$ exists and is continuous. Also there exists a positive constant M_k such that $\mathbb{E}\|K_1(t)\|_H^2\leq M_k$.

(H_7) The nonlinear function $\mathfrak{h}$ satisfies the Lipschitz condition and there exists a positive constants $M_{\mathfrak{h}},L_{\mathfrak{h}}>0$ such that

$$\int_Z \|\mathfrak{h}(t,x_1,\eta)-\mathfrak{h}(t,x_2,\eta)\|^2\lambda(d\eta)dt\leq M_{\mathfrak{h}}\|x_1-x_2\|_{\mathcal{B}}^2,$$
$$\int_Z \|\mathfrak{h}(t,x_1,\eta)-\mathfrak{h}(t,x_2,\eta)\|^4\lambda(d\eta)dt\leq L_{\mathfrak{h}}\|x_1-x_2\|_{\mathcal{B}}^4,$$

for all $x_1, x_2 \in \mathcal{B}$ and $t \in J$.

Under the above hypotheses, for each $v \in \mathcal{L}_2(0, b; H)$ there exists a unique mild solution

$$\begin{aligned} x_t(v) &= C_q(t)(\phi(0) + \mathbf{m}(x_{t_1}, ..., x_{t_m})(0)) + S_q(t)(\xi - \mathbf{n}(x)) \\ &+ \int_0^t T_q(t-s) f\Big(s, x_s, \int_0^s \sigma(s, \tau, x_\tau) d\tau\Big) ds + \int_0^t T_q(t-s) Bu(s) ds \\ &+ \int_0^t T_q(t-s)\Big[\int_{-\infty}^s g(\tau, x_\tau) d\omega(\tau)\Big] ds + \int_0^t \int_Z T_q(t-s)\mathfrak{h}(s, x_s, \eta)\widetilde{N}(ds, d\eta) \\ &+ \sum_{0<t_k<t} T_q(t-t_k) I_k(x_{t_k}) + \sum_{0<t_k<t} T_q(t-t_k)\overline{I}_k(x_{t_k}), k = \overline{1,n}. \end{aligned} \quad (3)$$

The solution mapping Υ from $\mathcal{L}_2(J; H)$ to $C(J; \mathcal{L}_2(\Omega, H))$ can be defined by $\Upsilon(v)(t) = x_t(v)(\cdot)$. Now we define the continuous linear operator Θ from $\mathcal{L}_2(J; H)$ to $\mathcal{L}_2(\Omega, H)$ by

$$\Theta p = \int_0^b T_q(b-s)p(s)ds, \quad \text{for} \quad p \in \mathcal{L}_2(J; H).$$

Definition 2.9. *Let the reachable set of the system (1) at time b be $\mathcal{R}_b(f, g, \mathfrak{h}) = \{x_b(Bu); \quad u \in \mathcal{L}_2(J; U)\}$, where $x_t(Bu)$ is a mild solution which satisfies with $v = Bu$.*

Definition 2.10. *The system (1) is said to be approximately controllable on J if $\overline{\mathcal{R}_b(f, g, \mathfrak{h})} = \mathcal{L}_2(\Omega, H)$, where $\overline{\mathcal{R}_b(f, g, \mathfrak{h})}$ is the closure of $\mathcal{R}_b(f, g, \mathfrak{h})$. Therefore, for every $\epsilon > 0$ and $\Psi \in D(A)$ there exists a control $u \in \mathcal{L}_2(J; U)$ such that*

$$\begin{aligned} &\mathbb{E}\|\Psi - C_q(t)\Big(\phi(0) + \mathbf{m}(x_{t_1}, ..., x_{t_m})(0)\Big) - S_q(t)(\xi - \mathbf{n}(x))) \\ &- \int_0^t T_q(t-s)\Big[f\Big(s, x_s(Bu), \int_0^s \sigma(s, \tau, x_\tau) d\tau\Big)\Big] ds \\ &- \int_0^t T_q(t-s)\Big[\int_{-\infty}^s g(\tau, x_s(Bu)) d\omega(s)\Big] ds - \int_0^t \int_Z T_q(t-s)\mathfrak{h}(s, x_s(Bu), \eta)\widetilde{N}(ds, d\eta) \\ &- \sum_{0<t_k<t} T_q(t-t_k) I_k(x_{t_k}) - \sum_{0<t_k<t} T_q(t-t_k)\overline{I}_k(x_{t_k}) - \Theta(Bu(s))\|^2 < \epsilon, \end{aligned}$$

where $x_t(Bu)$ is a solution of (1) at time t.

Now, we introduce the following assumptions (see [25, 32]). For any given $\epsilon > 0$ and $p(\cdot) \in \mathcal{L}_2(J; H)$, there exists some $u(\cdot) \in \mathcal{L}_2(J; U)$ such that

($\mathbb{A}_1$) $\|\Theta p - \Theta Bu\|_H < \epsilon$,

($\mathbb{A}_2$) $\|Bu(\cdot)\|_{\mathcal{L}_2(J;H)} \leq q_1\|p(\cdot)\|_{\mathcal{L}_2(J;H)}$, where q_1 is a positive constant independent of $p(\cdot)$,

($\mathbb{A}_3$) the constant q_1 satisfies

$$\begin{aligned}&8q_1b^2M^2\overline{K}_3(Tr(Q)M_g + (M_{\mathfrak{h}} + \sqrt{L_{\mathfrak{h}}}))\\&\quad\times \exp\Big\{4b^2M^2\{\overline{K}_3(M_{f_1}l + M_{f_2}\overline{K}_3^*) + 2(M_K + Tr(Q)M_g\overline{K}_3) + 2\overline{K}_3(M_{\mathfrak{h}} + \sqrt{L_{\mathfrak{h}}})\}\Big\} < 1.\end{aligned}$$

First, we show the approximate controllability of the corresponding deterministic system of (1) with $g = 0$ and $\mathfrak{h} = 0$.

Lemma 2.11. *(see [32]) Under the Hypotheses* $(H_1) - (H_7)$ *and* $(\mathbb{P}_1)$, $\overline{\mathcal{R}_b(0,0)} = H$.

Proof. Since the domain $D(A)$ of the operator A is dense in H(see [31]), it is sufficient to prove that $D(A) \subset \overline{\mathcal{R}_b(0,0)}$, that is, for every given $\varepsilon > 0$ and $\Psi \in D(A)$ there exists a $u(\cdot) \in \mathcal{L}_2(J;U)$ such that

$$\mathbb{E}\|\Psi - \mathcal{K}(b;\phi,\xi) - \Theta(Bu)\|^2 < \epsilon, \tag{4}$$

where

$$\begin{aligned}\mathcal{K}(b;\phi,\xi) &= C_q(t)\Big(\phi(0) + \mathbf{m}(x_{t_1},...,x_{t_m})(0)\Big) + S_q(t)(\xi - \mathbf{n}(x)))\\&\quad + \sum_{0<t_k<t} T_q(t-t_k)I_k(x_{t_k}) + \sum_{0<t_k<t} T_q(t-t_k)\overline{I}_k(x_{t_k})\end{aligned}$$

Let $\Psi \in D(A)$, then $\Psi - \mathcal{K}(b;\phi,\xi) \in D(A)$. So there exists some $p \in C^1(J;H)$ such that $\widetilde{\Psi} = \int_0^b S_q(b-s)p(s)ds$, where $\widetilde{\Psi} = \Psi - \mathcal{K}(b;\phi,\xi)$. The assumption $(\mathbb{A}_1)$ implies that for any given $\epsilon > 0$, there exists a control function $u(\cdot) \in \mathcal{L}_2(J;U)$ such that

$$\mathbb{E}\|\widetilde{\Psi} - \Theta(Bu)\|^2 < \epsilon.$$

By putting the value of $\widetilde{\Psi}$, we obtain (4), since ϵ is arbitrary. We conclude that $\mathcal{R}_b(0,0) \subset D(A)$. The denseness of the domain $D(A)$ in H implies the approximate controllability of the corresponding system (1) with $g = 0$ and $\mathfrak{h} = 0$. □

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Lemma 2.12. *Let u_1 and u_2 be in $\mathcal{L}_2(J;U)$. Then under the Hypotheses $(H_1)-(H_7)$, the solution mapping $\Upsilon(Bu)(t) = x_t(Bu)(t)$ of (1) satisfies*

$$\mathbb{E}\|x_t(Bu_1) - x_t(Bu_2)\|^2 \leq 4b^2M^2 \exp\Big\{4b^2M^2\{\overline{K}_3(M_{f_1}l + M_{f_2}\overline{K}_3^*) + 2(M_k + Tr(Q)M_g\overline{K}_3) + 2\overline{K}_3(M_{\mathfrak{h}} + \sqrt{L_{\mathfrak{h}}})\}\Big\}\mathbb{E}\|Bu_1 - Bu_2\|^2_{\mathcal{L}_2(0,b;H)}.$$

Proof. Let $y(\cdot,\phi) : (-\infty, b] \to \mathcal{L}_2(\Omega, H)$ be the function defined by

$$y(t,\phi) = \begin{cases} \phi(t), & t \in J_0, \\ C_q(t)\Big(\phi(0) + \mathbf{m}(x_{t_1}, ..., x_{t_m})(0)\Big), & t \in J \end{cases}$$

Denote $y(t,\phi)$ by $y(t)$ with the continuous map $t \to y_t$. Next, for each $z \in C(J;\mathcal{L}_2(\Omega,H)), z(0) = 0$, we denote $\widetilde{z}$ the function defined by $\widetilde{z}(\theta) = 0$, for $\theta < 0$ and $\widetilde{z}(t) = z(t)$, for $0 \leq t \leq b$.
So, if $x(v)(t)$ satisfies (3), we can decompose it as $x(v)(t) = z(v)(t) + y(t)$, for $0 \leq t \leq b$, which implies that $x_t(v) = \widetilde{z}_t(v) + y_t$, for $0 \leq t \leq b$ and for each $v \in \mathcal{L}_2(J;H)$ and that the function $z(\cdot)$ satisfies

$$\begin{aligned} z(t) = S_q(t)(\xi - \mathbf{n}(x))) &+ \int_0^t T_q(t-s)\Big[f\Big(s, \widetilde{z}_s(v) + y_s, \int_0^s \sigma(s,\tau,\widetilde{z}_\tau(v) + y_\tau)d\tau\Big)\Big]ds \\ &+ \int_0^t T_q(t-s)\Big[\int_{-\infty}^s g(\tau, \widetilde{z}_s(v) + y_s)d\omega(s)\Big]ds \\ &+ \int_0^t T_q(t-s)\int_Z \mathfrak{h}(s, \widetilde{z}_s(v) + y_s, \eta)\tilde{N}(ds,d\eta) \\ &+ \int_0^t T_q(t-s)v(s)ds. \end{aligned}$$

Thus for each $u_1, u_2 \in \mathcal{L}_2(J;U)$, it is clear that for $0 \leq t \leq b$,

$$\begin{aligned} &\mathbb{E}\|x_t(Bu_1) - x_t(Bu_2)\|^2 \\ &= \mathbb{E}\|(\widetilde{z}_t(Bu_1) + y_t) - (\widetilde{z}_t(Bu_2) + y_t)\|^2, \\ &\leq \mathbb{E}\|\widetilde{z}_t(Bu_1) - \widetilde{z}_t(Bu_2)\|^2, \\ &\leq 4\Big\{\mathbb{E}\|\int_0^t T_q(t-s)\Big(\Big[f\Big(s, \widetilde{z}_s(Bu_1) + y_s, \int_0^s \sigma(s,\tau,\widetilde{z}_\tau(Bu_1) + y_\tau\Big) \\ &\quad - f\Big(s, \widetilde{z}_s(Bu_2) + y_s, \int_0^s \sigma(s,\tau,\widetilde{z}_\tau(Bu_2) + y_\tau)\Big)\Big]d\tau\Big)ds\|^2 \\ &\quad + \mathbb{E}\|\int_0^t T_q(t-s)[g(s,\widetilde{z}_s(Bu_1) + y_s) - g(s,\widetilde{z}_s(Bu_2) + y_s)]d\omega(s)\|^2 \end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\|\int_0^t T_q(t-s)\int_Z[\mathfrak{h}(s,\widetilde{z}_s(Bu_1)+y_s,\eta)-\mathfrak{h}(s,\widetilde{z}_s(Bu_2)+y_s,\eta)]\tilde{N}(ds,d\eta)\|^2\\
&+\mathbb{E}\|\int_0^t T_q(t-s)[Bu_1(s)-Bu_2(s)]ds\|^2\Big\},\\
\le\ & 4\Big\{M^2\mathbb{E}\|\int_0^t\Big[f\Big(s,\widetilde{z}_s(Bu_1)+y_s,\overline{K}_3(\widetilde{z}_s(Bu_1)+y_s)\Big)\\
&-f\Big(s,\widetilde{z}_s(Bu_2)+y_s,\overline{K}_3(\widetilde{z}_s(Bu_2)+y_s)\Big)\Big]ds\|^2+M^2b^2(2M_K+2Tr(Q)M_g\overline{K}_3)\\
&+b^2M^2\overline{K}_3(2M_{\mathfrak{h}}+2\sqrt{L_{\mathfrak{h}}})\Big\}\mathbb{E}\|x(Bu_1)-x(Bu_2)\|^2+4b^2M^2\mathbb{E}\|Bu_1-Bu_2\|^2_{\mathcal{L}_2(0,b;H)},\\
\le\ & 4\Big\{b^2M^2\overline{K}_3(M_{f_1}l+M_{f_2}\overline{K}_3^*)\}+M^2b^2(2M_K+2Tr(Q)M_g\overline{K}_3)\\
&+b^2M^2\overline{K}_3(2M_{\mathfrak{h}}+2\sqrt{L_{\mathfrak{h}}})\Big\}\mathbb{E}\|z(Bu_1)-z(Bu_2)\|^2+4b^2M^2\mathbb{E}\|Bu_1-Bu_2\|^2_{\mathcal{L}_2(0,b;H)},\\
\le\ & 4b^2M^2\Big\{\overline{K}_3(M_{f_1}l+M_{f_2}\overline{K}_3^*)+(2M_K+2Tr(Q)M_g\overline{K}_3)\\
&+\overline{K}_3(2M_{\mathfrak{h}}+2\sqrt{L_{\mathfrak{h}}})\Big\}\mathbb{E}\|z(Bu_1)-z(Bu_2)\|^2+4b^2M^2\mathbb{E}\|Bu_1-Bu_2\|^2_{\mathcal{L}_2(0,b;H)}.
\end{aligned}
$$

By Gronwall's inequality

$$
\begin{aligned}
&\mathbb{E}\|x_.(Bu_1)-x_.(Bu_2)\|^2\\
&\le 4b^2M^2\exp\Big\{4b^2M^2\{\overline{K}_3(M_{f_1}l+M_{f_2}\overline{K}_3^*)+2(M_K+Tr(Q)M_g\overline{K}_3)+2\overline{K}_3(M_{\mathfrak{h}}+\sqrt{L_{\mathfrak{h}}})\}\Big\}\\
&\quad\times\mathbb{E}\|Bu_1-Bu_2\|^2_{\mathcal{L}_2(0,b;H)}.
\end{aligned}
$$
□

Theorem 2.13. *Under the Hypotheses* $(H_1)-(H_7)$ *and* $(\mathbb{A}_1)-(\mathbb{A}_3)$, *then the system (1) is approximately controllable on* J.

Proof. Since by Lemma 2.11, $\overline{\mathcal{R}_b(0,0)}=H$, to prove the approximate controllability of the system (1) , it is sufficient to show that $\overline{\mathcal{R}_b(0,0)}\subset\overline{\mathcal{R}_b(g,h)}$. For this, let us take $\Psi\in\overline{\mathcal{R}_b(0,0)}$, then, for any given $\epsilon>0$, there exists a control function $u\in\mathcal{L}_2(J;U)$ such that

$$\mathbb{E}\|\Psi-\mathcal{K}(b;\phi,\xi)-\Theta(Bu)\|^2<\frac{\epsilon}{2^5}. \tag{5}$$

Assume that $u_1\in\mathcal{L}_2(J;U)$ be arbitrary given. By hypothesis $(\mathbb{A}_2)$, there exists some $u_2\in\mathcal{L}_2(J;U)$ such that

$$
\begin{aligned}
&\mathbb{E}\|\Theta Bu-\int_0^b T_q(b-s)g(s,x_s(Bu_1))d\omega(s)\\
&\qquad-\int_0^b T_q(b-s)\int_Z\mathfrak{h}(s,x_s(Bu_1),\eta)\tilde{N}(ds,d\eta)-\Theta Bu_2\|^2<\frac{\epsilon}{2^5}.
\end{aligned}\tag{6}
$$

From (5) and (6), we have

$$\mathbb{E}\|\Psi - \mathcal{K}(b;\phi,\xi) - \int_0^b T_q(b-s)g(s,x_s(Bu_1))d\omega(s) - \int_0^b T_q(b-s)\int_Z \mathfrak{h}(s,x_s(Bu_1),\eta)\tilde{N}(ds,d\eta) - \Theta Bu_2\|^2 < \frac{\epsilon}{2^4}.$$

For $u_2 \in \mathcal{L}_2(J;U)$ thus obtained, we determine $w_2 \in \mathcal{L}_2(J;U)$ by hypotheses $(\mathbb{A}_1)$ and $(\mathbb{A}_2)$ such that

$$\mathbb{E}\|\int_0^b T_q(b-s)[g(s,x_s(Bu_2)) - g(s,x_s(Bu_1))]d\omega(s) + \int_0^b T_q(b-s) \times \int_Z [\mathfrak{h}(s,x_s(Bu_2),\eta) - \mathfrak{h}(s,x_s(Bu_1),\eta)]\tilde{N}(ds,d\eta) - \Theta Bw_2\|^2 < \frac{\epsilon}{2^5},$$

also, by $(\mathbb{A}_2)$ and Lemma 2.12, we have

$$\begin{aligned}\mathbb{E}\|Bw_2\|^2_{\mathcal{L}_2(J;H)} &\leq 2q_1\Big\{Tr(Q)\int_0^b \mathbb{E}\|g(s,x_s(Bu_2)) - g(s,x_s(Bu_1))\|^2 ds \\ &\quad + \mathbb{E}\|\int_0^b\int_Z [\mathfrak{h}(s,x_s(Bu_2),\eta) - \mathfrak{h}(s,x_s(Bu_1),\eta)]\tilde{N}(ds,d\eta)\|^2\Big\}, \\ &\leq 2q_1 b^2\overline{K}_3\Big\{Tr(Q)M_g + (M_{\mathfrak{h}} + \sqrt{L_{\mathfrak{h}}})\Big\}\mathbb{E}\|x_t(Bu_2) - x_t(Bu_1)\|^2, \\ &\leq 8q_1 b^2 M^2\overline{K}_3(Tr(Q)M_g + (M_{\mathfrak{h}} + \sqrt{L_{\mathfrak{h}}}))\exp\Big\{4b^2M^2\{\overline{K}_3(M_{f_1}l + M_{f_2}\overline{K}_3^*) \\ &\quad + 2(M_K + Tr(Q)M_g\overline{K}_3) + 2\overline{K}_3(M_{\mathfrak{h}} + \sqrt{L_{\mathfrak{h}}})\}\Big\}\mathbb{E}\|Bu_1 - Bu_2\|^2_{\mathcal{L}_2(0,b;H)}.\end{aligned}$$

Define $u_3 = u_2 - w_2$ in $\mathcal{L}_2(J;U)$, which has the following property

$$\begin{aligned}&\mathbb{E}\|\Psi - \mathcal{K}(b;\phi,\xi) - \int_0^b T_q(b-s)g(s,x_s(Bu_2))d\omega(s) - \int_0^b T_q(b-s) \\ &\times \int_Z \mathfrak{h}(s,x_s(Bu_2),\eta)\tilde{N}(ds,d\eta) - \Theta Bu_3\|^2 \leq 2\Big\{\mathbb{E}\|\Psi - \mathcal{K}(b;\phi,\xi) - \int_0^b T_q(b-s) \\ &\times g(s,x_s(Bu_1))d\omega(s) - \int_0^b T_q(b-s)\int_Z \mathfrak{h}(s,x_s(Bu_1),\eta)\tilde{N}(ds,d\eta) - \Theta Bu_2\|^2 \\ &+ \mathbb{E}\|\int_0^b T_q(b-s)[g(s,x_s(Bu_2)) - g(s,x_s(Bu_1))]d\omega(s) + \int_0^b T_q(b-s) \\ &\times \int_Z [\mathfrak{h}(s,x_s(Bu_2),\eta) - \mathfrak{h}(s,x_s(Bu_1),\eta)]\tilde{N}(ds,d\eta) - \Theta Bw_2\|^2\Big\} \leq \Big(\frac{1}{2^3} + \frac{1}{2^4}\Big)\epsilon.\end{aligned}$$

By mathematical induction method, there exists a sequence $u_n(\cdot) \in \mathcal{L}_2(J;U)$ such that

$$\mathbb{E}\|\Psi - \mathcal{K}(b;\phi,\xi) - \int_0^b T_q(b-s)g(s,x_s(Bu_n))d\omega(s) - \int_0^b T_q(b-s)$$

$$\times\int_Z \mathfrak{h}(s,x_s(Bu_n),\eta)\tilde{N}(ds,d\eta)-\Theta Bu_{n+1}\|^2\leq\Big(\frac{1}{2^3}+\cdots+\frac{1}{2^{n+2}}\Big)\epsilon,\quad n=1,2,\ldots$$

$$\begin{aligned}\mathbb{E}\|Bu_{n+1}-Bu_n\|^2_{\mathcal{L}_2(J;H)}\leq\ & 8q_1b^2M^2\overline{K}_3(Tr(Q)M_g+(M_{\mathfrak{h}}+\sqrt{L_{\mathfrak{h}}}))\\ &\times\exp\Big\{4b^2M^2\{\overline{K}_3(M_{f_1}l+M_{f_2}\overline{K}_3^*)+2(M_K+Tr(Q)M_g\overline{K}_3)\\ &+2\overline{K}_3(M_{\mathfrak{h}}+\sqrt{L_{\mathfrak{h}}})\}\Big\}\mathbb{E}\|Bu_n-Bu_{n-1}\|^2_{\mathcal{L}_2(J;H)}.\end{aligned}$$

Clearly, by hypothesis $(\mathbb{A}_3)$, the sequence $\{Bu_n;n=1,2,\ldots\}$ is a Cauchy sequence in Hilbert space $\mathcal{L}_2(J;H)$ and there exists some $v\in\mathcal{L}_2(J;H)$ such that $\lim_{n\to\infty}Bu_n=v$ in $\mathcal{L}_2(J;H)$. Therefore, for any given $\epsilon>0$, there exists some integer N_ϵ such that $\mathbb{E}\|\Theta Bu_{N_\epsilon+1}-\Theta Bu_{N_\epsilon}\|^2<\frac{\epsilon}{2^2}$,

$$\begin{aligned}&\mathbb{E}\|\Psi-\mathcal{K}(b;\phi,\xi)-\int_0^b T_q(b-s)g(s,x_s(Bu_{N_\epsilon}))d\omega(s)-\int_0^b T_q(b-s)\int_Z\mathfrak{h}(s,x_s(Bu_{N_\epsilon}),\eta)\\ &\times\tilde{N}(ds,d\eta)-\Theta Bu_{N_\epsilon}\|^2\leq 2\Big\{\mathbb{E}\|\Psi-\mathcal{K}(b;\phi,\xi)-\int_0^b T_q(b-s)g(s,x_s(Bu_{N_\epsilon}))d\omega(s)\\ &-\int_0^b T_q(b-s)\int_Z\mathfrak{h}(s,x_s(Bu_{N_\epsilon}),\eta)\tilde{N}(ds,d\eta)-\Theta Bu_{N_\epsilon+1}\|^2+\mathbb{E}\|\Theta Bu_{N_\epsilon+1}-\Theta Bu_{N_\epsilon}\|^2\\ &\leq 2\Big(\frac{1}{2^3}+\ldots+\frac{1}{2^{N_\epsilon+1}}\Big)\epsilon+2\Big(\frac{1}{2^2}\Big)\epsilon.\end{aligned}$$

This means that $\Psi\in\overline{\mathcal{R}_b(g,h)}$. Thus $\overline{\mathcal{R}_b(0,0)}\subset\overline{\mathcal{R}_b(g,h)}$, hence the nonlinear system (1) is approximately controllable on J. □

3. Application

Consider the following stochastic fractional system in the form

$$\begin{aligned}{}^cD_t^q z(t,\zeta)=\ &\frac{\partial^2}{\partial x^2}z(t,\zeta)+a_0(t,\zeta)z(t,\zeta)+k_1(t,z(t,\zeta))+\int_{-\infty}^{t}a_2(s-t)z(s,\zeta)ds\\ &+\int_{-\infty}^{t}b_2(t)b_3(s-t)z(s,\zeta)d\omega(s)+\int_Z\eta\Big(\int_{-\infty}^{t}a_3(s-t)z(s,\zeta)ds\Big)\tilde{N}(dt,d\eta)\end{aligned}$$

$$z(t,0)=z(t,\pi)=0,\quad z'(t,0)=z'(t,\pi)=0,\quad t\in J,$$

$$z(t,\zeta)=\phi(t,\zeta),\quad z'(t,\zeta)=\xi(t,\zeta),\quad t\in(-\infty,0],$$

$$z(t,\zeta)+\Big(\sum_{i=1}^m C_i z(t_i+t,\zeta)\Big)(t)=\zeta_0(t),\quad 0<\zeta<\pi,$$

$$z'(t,0) = \zeta'(0) + \sum_{i=1}^{m} D_i\zeta, \quad 0 < \zeta < \pi,$$

$$[z(t_k^+) - z(t_k^-)]\zeta = I_k(z(t_k))\zeta = \int_0^\pi \widehat{c}_k(\zeta, z(s,t_k))ds, \quad k = 1,2,...,n,$$

$$[z'(t_k^+) - z'(t_k^-)]\zeta = \overline{I}_k z(t_k))\zeta = \int_0^\pi \widehat{d}_k(\zeta, z(s,t_k))ds, \quad k = 1,2,...,n, \tag{7}$$

where ${}^cD_t^q$ is the Caputo fractional partial derivative of order $1 < q < 2$. $\widehat{c}_k$, $\widehat{d}_k$ are continuous for $k = 1,2,..,n$ and C_i, D_i, $i = 1,2,...,m$ are fixed numbers. Define the operator $\mathbf{m} : H^m \to H$ by $\mathbf{m}(z_{t_1}, z_{t_2}, ..., z_{t_m})(t)=\sum_{i=0}^{m} C_i z(t_i + t), x \in \mathcal{B}$, $t \in (-\infty, 0]$ and $\|\mathbf{m}(\cdot)\|_H^2 \le G$. I_k and $\overline{I}_k : H \to H$ are appropriate functions. Let $0 = t_0 < t_1 < t_2 < ... < t_n < b$ be the given time points and the symbol $\Delta z(t)$ represents the jump of the function z at t defined by $\Delta z(t) = z(t^+) - z(t^-)$. Also $\omega(t)$ denotes a standard one-dimensional Wiener process defined on a stochastic basis $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}, \mathbb{P})$. To write the above system (7) into the abstract form (1), we can choose the space $H = \mathcal{L}_2([0,\pi])$. $\mathcal{B}$ is the phase space. Define $A : D(A) \subseteq H \to H$ by $Ax = x''$, with the domain $D(A) = \{x \in H : x, x'$ are absolutely continuous, $x'' \in H$ and $x(0) = x(\pi) = 0\}$. In particular $Ax=\sum_{n=1}^{\infty} -n^2\langle x, e_n\rangle e_n$, $x \in D(A)$. Then A is the infinitesimal generator of a strongly continuous cosine family $C(t)x=\sum_{n=1}^{\infty} cos(nt)\langle x, e_n\rangle e_n$, $t \in \mathbb{R}$ and the associated sine family in H defined by $S(t)x=\sum_{n=1}^{\infty} \frac{sin(nt)}{n}\langle x, e_n\rangle e_n$, $t \in \mathbb{R}$. From [33], for all $x \in H$, $t \in \mathbb{R}$, $\|S(t)\| \le 1$ and $\|C(t)\| \le 1$. Hence A consists of eigen values n for $n \in \mathbb{N}$, with associated eigen vectors $e_n(x)=\sqrt{\frac{2}{\pi}} sin(nx)$, $n \in \mathbb{N}$.

For $(t, \varphi) \in J \times \mathcal{B}$, we have $\varphi(\theta)(\cdot) = \phi(\theta, \cdot)$ and $\varphi'(\theta)(\cdot) = \xi(\theta, \cdot)$, $(\theta, \cdot) \in (-\infty, 0] \times [0, \pi]$. Let $z(t)(\cdot) = z(t, \cdot)$. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space and $\{K(t) : t \in J\}$ is a Poisson point process taking values in the space $K = [0, \infty)$ with a σ-finite intensity measure $\lambda(dy)$. The Poisson counting measure $\widetilde{N}(dt, dy)$ is induced by $K(\cdot)$ and the compensating martingale measure is denoted by $\widetilde{N}(dt, dy):=N(dt, dy) - dt\lambda(dy)$.

For $q \in (1, 2)$, A generates a strongly continuous cosine family $C(t)$, it follows from the subordinate principle (see Theorem 3.1, [7]) that A generates a strongly continuous exponentially bounded fractional cosine family $C_q(t)$ such that $C_q(0) = I$ and

$$C_q(t) = \int_0^\infty \varphi_{t,\frac{q}{2}}(s)C(s)ds, \quad t > 0,$$

$$\text{where} \quad \varphi_{t,\frac{q}{2}}(s) = t^{-\frac{q}{2}}\varrho_{\frac{q}{2}}(st^{-\frac{q}{2}}) \quad \text{and}$$

$$\varrho_{\vartheta}(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma(-\vartheta n + 1 - \theta)} \quad , \quad (0 < \theta < 1).$$

Assume that the continuous functions $f : J \times \mathcal{B} \to H$, $k_1 : J \times J \times \mathcal{B} \to H$, $g : J \times \mathcal{B} \to \mathcal{L}_Q(K, H)$, $\mathfrak{h} : J \times \mathcal{B} \times Z \to H$, I_k and $\overline{I}_k : H \to H$ are defined by

$$f(t, \varphi, \psi)(\zeta) = k_1(t, \varphi(0,)) + \int_{-\infty}^{t} a_2(s)\varphi(0, \zeta)ds, \quad t \geq 0, \zeta \in [0, \pi],$$

$$g(t, \varphi)(\zeta) = \int_{-\infty}^{t} b_2(t)b_3(s - t)\varphi(s, \zeta)d\omega(s), \quad t \geq 0, \zeta \in [0, \pi],$$

$$\mathfrak{h}(t, \varphi)(\zeta) = \int_{-\infty}^{t} a_3(s - t)\varphi(s, \zeta)ds \quad t \geq 0, \zeta \in [0, \pi],$$

$$I_k(\varphi)(\zeta) = \int_0^{\pi} \widehat{c}_k(\zeta, \varphi(s, 0))ds, \quad k = 1, 2, ..., n, \zeta \in [0, \pi],$$

$$\overline{I}_k(\varphi)(\zeta) = \int_0^{\pi} \widehat{d}_k(\zeta, \varphi(s, 0))ds, \quad k = 1, 2, ..., n, \zeta \in [0, \pi].$$

We have the following assumptions

(A1) The function $\zeta \to a_0(t, \zeta)$ is continuous for each $t \in \mathbb{R}$ with $a_0(t, \zeta) \leq -\delta_0(\delta_0 > 0)$ for all $\zeta \in [0, \pi]$.

(A2) The function $k_1 : \mathbb{R}^3 \to \mathbb{R}$ is continuous there is a continuous and integrable function $m : \mathbb{R} \to [0, \infty)$ such that

$$|k_1(t, x, y)| \leq m(t)(|x| + |y|), t \geq 0, (t, x, y) \in \mathbb{R}^3.$$

(A3) The function $a_2 : \mathbb{R}^4 \to \mathbb{R}$ is completely continuous and there is a continuous function $a : \mathbb{R}^2 \to \mathbb{R}$ such that

$$|a_2(t, s, x, y)| \leq a(t, s)|x|, (t, s, x, y) \in \mathbb{R}^4.$$

(A4) Also the function $a_2(t, s, x, y) : \mathbb{R}^4 \to \mathbb{R}$ is continuous, strongly measurable and $|a(t, s)|^2 \leq \gamma_1\alpha(s)$ for an integrable function $\alpha : \mathbb{R} \to [0, \infty)$ and a constant γ_1 such that

$$\mathbb{E}\|a_2(t, s, x, y)\|_H^2 \leq \gamma_1\alpha(s)\Omega_1(\|x\|), \quad \liminf_{\tau\to\infty} \frac{\Omega_1(\tau)}{\tau} = \upsilon_1 < \infty.$$

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(A5) There exists a continuous functions $m_1 : J \to [0, \infty)$ and a continuous non-decreasing function $W : [0, \infty) \to (0, \infty)$ such that for every $(t, \varphi, y) \in J \times \mathcal{B} \times H$, we have

$$\mathbb{E}\|f(t,\varphi,y)\|_H^2 \le m_1(t)W(\|\varphi\| + \|y\|), \quad \liminf_{\tau\to\infty} \frac{W(\tau)}{\tau} = \upsilon_0 < \infty.$$

(A6) Using example 1.1, we have

$$\begin{aligned}
\mathbb{E}\|f(t,\varphi_1) - f(t,\varphi_2)\|_H^2 &= \mathbb{E}\left[\left(\int_0^\pi \left|\int_{-\infty}^0 a_2(t,s,\varphi_1(s),y)ds - a_2(t,s,\varphi_2(s),y)ds\right|^2\right] dx\right)^{\frac{1}{2}}\Bigg]^2 \\
&\le \mathbb{E}\left[\left(\int_0^\pi \left(\int_{-\infty}^0 a(t,s)|\varphi_1(s)-\varphi_2(s)|ds\right)^2 dx\right)^{\frac{1}{2}}\right]^2 \\
&\le \mathbb{E}\left[\left(\int_{-\infty}^0 \frac{(a(t,s))^2}{\widetilde{h}(s)}ds\right)^{\frac{1}{2}}\left(\int_{-\infty}^0 \widetilde{h}(s)\|\varphi_1(s)-\varphi_2(s)\|^2 ds\right)^{\frac{1}{2}}\right]^2 \\
&\le M_f\left[\|\varphi(0)\| + \left(\int_{-\infty}^0 \widetilde{h}(s)\|\varphi_1(s)-\varphi_2(s)\|^2 ds\right)^{\frac{1}{2}}\right]^2 \\
&\le M_f\|\varphi_1 - \varphi_2\|_{\mathcal{B}}^2,
\end{aligned}$$

for all (t,φ_1),$(t,\varphi_2) \in J \times \mathcal{B}$, where $M_f = \left(\int_{-\infty}^0 \frac{(a(t,s))^2}{\widetilde{h}(s)}ds\right)^{\frac{1}{2}} < \infty$.

(A7) For all $(t,\varphi) \in J \times \mathcal{B}$, we have

$$\begin{aligned}
\mathbb{E}\|g(t,\varphi)\|_H^2 &= \mathbb{E}\left[\left(\int_0^\pi \left|\int_{-\infty}^0 b_2(t)b_3(s)\varphi(s)(x))ds\right|^2\right] dx\right)^{\frac{1}{2}}\Bigg]^2 \\
&\le \mathbb{E}\left[\|b_2\|_\infty\left(\int_0^\pi \left(\int_{-\infty}^0 b_3(s)|\varphi(s)|ds\right)^2 dx\right)^{\frac{1}{2}}\right]^2 \\
&\le \mathbb{E}\left[\|b_2\|_\infty\left(\int_{-\infty}^0 \frac{(b_3(t,s))^2}{\widetilde{h}(s)}ds\right)^{\frac{1}{2}}\left(\int_{-\infty}^0 \widetilde{h}(s)\|\varphi(s)\|^2 ds\right)^{\frac{1}{2}}\right]^2 \\
&\le M_g\left[\|\varphi(0)\| + \left(\int_{-\infty}^0 \widetilde{h}(s)\|\varphi(s)\|^2 ds\right)^{\frac{1}{2}}\right]^2 \\
&\le M_g\|\varphi\|_{\mathcal{B}}^2,
\end{aligned}$$

where $M_g = \left(\int_{-\infty}^0 \frac{(b_3(t,s))^2}{\widetilde{h}(s)}ds\right)^{\frac{1}{2}} < \infty$.

(A8) Also there exists continuous function $m_2 : J \to [0,\infty)$ and continuous non-decreasing function $\Omega_2 : [0,\infty) \to (0,\infty)$ such that for every $(t,\varphi) \in J \times \mathcal{B}$, we have

$$\mathbb{E}\|\mathfrak{h}(t,\varphi,\eta)\|_H^2 \le m_2(t)\Omega_2(\|\varphi\|^2), \quad \liminf_{\tau\to\infty} \frac{\Omega_2(\tau)}{\tau} = \upsilon_3 < \infty,$$

(A9) The function $\widehat{c}_k : \mathbb{R} \to \mathbb{R}$ is continuous such that $M_{\widehat{c}_k} = \left(\int_{-\infty}^0 \frac{(\widehat{c}_k(\zeta,\varphi(s,0)))^2}{\widetilde{h}(s)} ds\right)^{\frac{1}{2}} < \infty$, for every $k = 1, 2 \dots n$.

(A10) The function $\widehat{d}_k : \mathbb{R} \to \mathbb{R}$ is continuous such that $M_{\widehat{d}_k} = \left(\int_{-\infty}^0 \frac{(\widehat{d}_k(\zeta,\varphi(s,0)))^2}{\widetilde{h}(s)} ds\right)^{\frac{1}{2}} < \infty$, for every $k = 1, 2 \dots n$.

Moreover, the maps f, g, $\mathfrak{h}$,I_k and $\overline{I}_k$ are bounded linear operators such that $\|f\|^2 \le M_f$, $\|g\|^2 \le M_g$, $\|\mathfrak{h}\|^2 \le N_{\mathfrak{h}}$, $\|I_k\|^2 \le M_{\widehat{c}_k}$ and $\|\overline{I}_k\|^2 \le M_{\widehat{d}_k}$
Now for $q = 2$, we have

$$\begin{aligned}
\|AT_q(t-s)\|^2 &\le \|A\mathcal{J}_t^{q-1}C_q(t-s)\|^2 \\
&\le \|\frac{d^2}{dt^2}\mathcal{J}_t^{q-1}C_q(t-s)\|^2 \\
&\le \|\frac{d^2}{dt^2}\mathcal{J}_t^{2-(3-q)}C_q(t-s)\|^2 \\
&\le \|{}^cD_t^{3-q}C_q(t)\|^2 \\
&\le \|{}^cD_tC_2(t)\|^2 \\
&\le \|\varphi_{t,1}(s)C(s)\|^2 \\
&\le \|\varphi_{t,1}(s)\|^2 = \|t^{-1}\varrho_1(-st^{-1})\|^2 = \left\|\frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^n s^n}{t^n n!\Gamma(1-(n+\theta))}\right\|^2, \qquad (0<\theta<1).
\end{aligned}$$

It is clear that the fractional differential system (2) is approximately controllable on J for $q = 2$. Hence for $q = 2$ with the above choices, the system (7) can be rewritten to the abstract form (1) and all the conditions of Theorem 2.11 and Theorem 2.12 are satisfied. Thus there exists mild soluti8ns for the system (7). Moreover all the conditions of Theorem 2.13 are satisfied and hence the fractional stochastic differential system with Poisson jumps (7) is approximately controllable on J.

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